In a nutshell: Neumann and insulated boundary conditions

Given a second order linear ordinary differential equation with constant coefficients

$$a_2(x)u^{(2)}(x) + a_1(x)u^{(1)}(x) + a_0(x)u(x) = g(x),$$

two spatial boundary points [a, b] and two boundary conditions: either Dirichlett, $u(a) = u_a$ and $u(b) = u_b$; or Neumann: $u^{(1)}(a) = u_a^{(1)}$ and $u^{(1)}(b) = u_b^{(1)}$. Recall that if a constant has a derivative, that indicates that the constant represents a slope; it does not suggest we are taking the derivative of that constant. An insulated boundary condition is when we have a Neumann condition and the derivative is set to zero.

Parameters:

n The number of sub-intervals into which [a, b] will be divided.

1. Set $h \leftarrow \frac{b-a}{n}$ and $x_k \leftarrow a + kh$ noting that $x_n = b$.
2. For k going from 1 to $n - 1$, assign the following:
$$\begin{aligned} p_k &\leftarrow 2a_2(x_k) - a_1(x_k)h \\ q_k &\leftarrow -4a_2(x_k) + 2a_0(x_k)h^2 \\ r_k &\leftarrow 2a_2(x_k) + a_1(x_k)h \end{aligned}$$
If the linear ordinary differential equation has constant coefficients, all p, q and r values are equal to each other.
3. Create the system of $n - 1$ linear equations in $n - 1$ unknowns
$$A\mathbf{u} = \begin{pmatrix} q_1 & r_1 & & & & & \\ p_2 & q_2 & r_2 & & & & \\ & p_3 & q_3 & r_3 & & & \\ & & p_4 & q_4 & r_4 & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & p_{n-2} & q_{n-2} & r_{n-2} \\ & & & & & p_{n-1} & q_{n-1} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ \vdots \\ u_{n-2} \\ u_{n-1} \end{pmatrix} = \begin{pmatrix} 2g(x_1)h^2 \\ 2g(x_2)h^2 \\ 2g(x_3)h^2 \\ 2g(x_4)h^2 \\ \vdots \\ 2g(x_{n-2})h^2 \\ 2g(x_{n-1})h^2 \end{pmatrix} = \mathbf{g}$$
4. We modify the matrix A and the vector **g** as follows:
   a. If the left-hand boundary condition is a Dirichlet condition, update $g_1 \leftarrow g_1 - p_1 u_a$;
   b. otherwise, the left-hand boundary condition is a Neumann condition, so update:
      i. $a_{1,1} \leftarrow a_{1,1} + \frac{4}{3}p_1$
      ii. $a_{1,2} \leftarrow a_{1,2} - \frac{1}{3}p_1$
      iii. $g_1 \leftarrow g_1 + \frac{2}{3}p_1 u_a^{(1)} h$

c. If the right-hand boundary condition is a Dirichlet condition, update $g_{n-1} \leftarrow g_{n-1} - r_{n-1}u_b$;

d. otherwise, the right-hand boundary condition is a Neumann condition, so update:

i. $a_{n-1,n-2} \leftarrow a_{n-1,n-2} - \frac{1}{3}r_{n-1}$

ii. $a_{n-1,n-1} \leftarrow a_{n-1,n-1} + \frac{4}{3}r_{n-1}$

iii. $g_{n-1} \leftarrow g_{n-1} - \frac{2}{3}r_{n-1}u_b^{(1)}h$

Note that the right-hand vector **g** is not updated for insulated boundary conditions.

5. Solve the updated system of linear equations $A\mathbf{u} = \mathbf{g}$.
6. The approximation of $u(x_k)$ is u_k for $k = 1, \ldots, n - 1$; and
   a. If the left-hand boundary condition is Dirichlet, $u(x_0) = u(a) = u_a$;
   b. otherwise, the left-hand boundary condition is Neumman, so

$$u(x_0) = u(a) \approx -\frac{2}{3}u_a^{(1)}h + \frac{4}{3}u_1 - \frac{1}{3}u_2$$

   c. If the right-hand boundary condition is Dirichlet, $u(x_n) = u(b) = u_b$;
   d. otherwise, the right-hand boundary condition is Neumann, so

$$u(x_n) = u(b) \approx \frac{2}{3}u_b^{(1)}h + \frac{4}{3}u_{n-1} - \frac{1}{3}u_{n-2}$$